Renormalization group theory for perturbed evolution equations

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We show that proto-RG (renormaliation group) theory can be used to give a systematic description of the evolution of soltion of perturbed equations. The equations describing the deformation of its shape as the effect of perturbation are proto-RG equations. The RG approach may be simpler than inverse scattering theory (IST) and another approaches, because it dose not rely on any knowledge of IST. It is very concise and easy to understand.

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I. INTRODUCTION

The standard soliton equations (KdV, NLS and so on) are highly idealized. In more realistic situations, it is important to understand nonlinear evolution equations under the influence of perturbations [1]. Several different approaches to the analytic description of soliton dynamics in these perturbed systems are known. The most powerful method to deal with these cases is based on IST (inverse scattering theory) [2,3]. The basic idea of IST is to represent a nonlinear evolution equation for a function $u(x,t)$ in the form of the so-called Lax pair $(\hat{L},\ \hat{A})$, $\hat{L}_t+[\hat{L},\ \hat{A}]=0$, where $\hat{L}$ and $\hat{A}$ are some linear operators with coefficients dependent on the function $u(x,t)$. First, we can solve the direct scattering problem, i.e., solving the auxiliary linear equations $\hat{L}\Psi=\lambda\Psi$ and $\Psi_t=\hat{A}\Psi$. From the first equation we can find the scattering data, and from the second equation, we can find the temporal evolution of the scattering data. Then we can construct the function $u(x,t)$ on the basis of the scattering data, i.e., solve the inverse scattering problem. The whole process is effective for integrable systems and it turns out that for any nonlinear evolution equation that is reasonably close to a nonlinear evolution equation that can be exactly solved by IST, the total evolution of the scattering data can be given and also be expanded in a perturbation expansion. That is to say that we can perform a perturbation theory for the above two auxiliary linear equations. Thus one can determine the effects of perturbations such as small dissipations, relaxations, etc., on the evolution of these scattering data, and further, the effects on the soliton states. However, it is inconvenient to use for one who is not familiar with IST.

Another alternative way to study soliton perturbations is the so-called direct perturbation theory based on a standard multiple scales procedure [4–6]. In this scheme, a basic technical ingredient is to linearize original nonlinear equations on the background of the unperturbed solution for taking a perturbation into account by rewriting them in terms of multiple scale variables ("fast" and "slow"). Then the time dependence of the soliton parameters and the first-order correction are readily available. The original work with important contribution to the direct method by constructing a Green's function process would be highly recommended [4].

In this paper, we perform a renormalization group (RG) approach [7] to the perturbed soliton equations and compare the result with other methods. One purpose of the present paper is to demonstrate that an analytic description of the evolution of a soliton, including the deformation of its shape, can be given, and the equations describing the evolution of the soliton amplitude and velocity as the effect of perturbation are RG equations. The apparent advantage of the RG approach is that the starting point is simply a straightforward naive perturbation expansion for which very little *a priori* knowledge is required. We also see that the RG approach may be more efficient and concise in practice than other methods.

We also notice that the so-called proto-RG scheme has been proposed recently [8,9]. The method does not need the explicit perturbation solutions that are required in the standard RG, so the process in this scheme is much simple than the original standard RG. It is very natural to use the proto-RG scheme to deal with these perturbed soliton equations. The present scheme has been illustrated with many examples that involve only one zero eigenfunction in the first-order equation [8,9]. However, for these perturbed soliton equations, the zero eigenfunctions of the first-order equation are usually degenerate, and we should consider some important information about the source of the secular terms in this case when we employ the proto-RG method. The other purpose of the our paper is to give some important examples that are treated very well in the present version of proto-RG.

In Secs. II, III, and IV, the details of proto-RG perturbative calculation are supplied for the perturbed time dependent Ginzbury-Landau (TDGL) equation, the perturbed KdV equation, and the perturbed NLS equation. Section V closes the paper with our conclusions and some expectations.

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II. PROTO-RG FOR THE PERTURBED TDGL EQUATION

We begin this section with the a simple description of proto-RG (See Refs. [8,9], for those who wish to know the details of the proto-RG method, the second Ref. [9] of Shiwa's should be a good starting point.) For a perturbed problem

$$\hat{L}(\partial_x)u=\varepsilon\hat{N}[u], \tag{1}$$

where x denotes all the independent spatial and/or temporal variables, the unperturbed equation and the first-order equation are

$$\hat{L}(\partial_x)u_0=0 \tag{2}$$

and

$$\hat{L}(\partial_x)u_1=\hat{N}_1[u_0]. \tag{3}$$

Here $\hat{L}$ is a linear differential operator, $\hat{N}$ is generally a nonlinear operator and $\hat{N}_1$ is the εth order terms of $\hat{N}$. We can write its perturbation result as

$$u=u_0(A)+\varepsilon u_1(x)+\cdots, \tag{4}$$

where A denotes all the constants such as amplitude or phase to be renormalized in the following. Here proto-RG scheme does not need the explicit expression of perturbation result, while the standard RG requires it. We introduce X and renormalized A_R and denote u_{1s} as the singular terms of u_1 with variables x replaced by X. Then the renormalized perturbation series reads

$$u=u_0(A_R)+\varepsilon[u_1(x)-u_{1s}(x,X)]+\cdots. \tag{5}$$

The proto-RG operator $\hat{S}$ can be constructed as follows: We split the differential operators ∂_x to $\partial_x+\partial_X$ in $\hat{L}$ and subtract the original operator to make $\hat{F}=\hat{L}(\partial_x+\partial_X)-\hat{L}(\partial_x)$. Then combine $\hat{F}$ with the operator $\hat{P}$ corresponding to the projection onto the null space of $\hat{L}$. Finally, identify $x=X$ in the result. The overall process defines the proto-RG operator $\hat{S}=\hat{P}\hat{F}$.

We apply $\hat{S}$ to u_{1s} and from the first-order equation (3), we can read the outcome as

$$\hat{S}u_{1s}=[\hat{N}_1[u_0]]_{\parallel}, \tag{6}$$

where $[\,]_{\parallel}$ denotes the projection onto the null space of $\hat{L}$. We can also apply $\hat{S}$ to renormalized result (5) and we obtain

$$\hat{S}u_0(A_R)=\varepsilon\hat{S}u_{1s}. \tag{7}$$

Therefore from the above two equations, we have

$$\hat{S}u_0(A_R)=\varepsilon[\hat{N}_1[u_0]]_{\parallel}. \tag{8}$$

After some simple reduction we can get the proto-RG equation of A_R. It is obviously from Eq. (8) that in this approach, if one knows the zero eigenfunction of $\hat{L}$, one can almost read off the final result by inspection.

Now we use the above method to deal with the perturbed TDGL equation

$$u_t-\frac{1}{2}u_{xx}-u+u^3=\varepsilon R[u]. \tag{9}$$

The TDGL equation has a single-soliton solution

$$u_0(x,t)=\tanh z,\quad z=x-\chi_0, \tag{10}$$

where χ_0 is the initial position.

To proceed, we transfer the variables from $\{t,x\}$ to $\{t,z\}$ and perform a εth perturbation to this equation,

$$u=u_0(\chi_0)+\varepsilon u_1(t,z)+\cdots. \tag{11}$$

In the first order, we find

$$[\partial_t-\hat{L}]u_1=\hat{R}_1[u_0], \tag{12}$$

where

$$\hat{L}=\frac{1}{2}\partial_{zz}+1-3u_0^2=\frac{1}{2}\partial_{zz}+3\ \text{sech}^2\ z-2 \tag{13}$$

is a self-adjoint operator and $\Phi_0(z)=\sqrt{3}/2\ \text{sech}^2\ z$ is the zero eigenfunction of $\hat{L}$ [10]. We can introduce the renormalized χ_R:

$$\chi_0=Z_1(\tau)\chi_R=\left[1+\sum_1^{\infty}a_n\varepsilon^n\right]\chi_R, \tag{14}$$

and the renormalized result to the first order is

$$u=u_0(\chi_R)+\varepsilon[u_1(t)-u_{1s}(t,\tau)]+\cdots, \tag{15}$$

where the renormalization constant a_1 is chosen as

$$-\chi_R a_1\,\text{sech}^2 z+u_{1s}(t,\tau)=0. \tag{16}$$

The reader may wonder why we neglect the introduction of Z to replace z in u_{1s}, i.e., the spatial singular behavior is omitted in this case. For partial differential equation, we should consider the spatial as well as temporal singular behavior generally. In Refs. [8] and [9], the authors discuss space-time behavior of numerous examples such as Swift-Hohenberg equation, etc. In those systems no spatial or temporal constraints are imposed, therefore secular terms can emerge in spatial as well as in temporal parts. However, in the evolution equations of the present paper, we always seek the solutions (of our interest) which vanish in infinity, i.e., this boundary condition restricts the systems bounded in the space and only the temporal singular behavior can appear in the solutions. Actually in other approaches for the perturbed evolution equations such as IST, the solutions are always supposed to have the same asymptotic behavior: when spatial coordinate goes to infinity, the solutions vanish. It is clear that in this case we can consider the temporal secular terms only. Now we find simply $\hat{F}=\partial_\tau$.

If we apply $\hat{S}=\hat{P}\hat{F}$ to u_{1s}, it must be identical to the coefficient of $\Phi_0(z)$ on the right-hand side (rhs) of the first-order equation (12), thus we have

$$\hat{S}u_{1s}=f_0, \tag{17}$$

where

$$f_0=\int_{-\infty}^{+\infty}\hat{R}_1[u_0]\Phi_0(z)dz, \tag{18}$$

i.e., this scalar product $\langle\Phi_0\cdot\hat{R}_1[u_0]\rangle$ realizes the needed projection in the proto-RG operator.

We now apply the proto-RG operator $\hat{S}$ on the renormalized perturbation result (15), then we obtain

$$\hat{S}(u_0(\chi_R))=\varepsilon\hat{S}u_{1s}. \tag{19}$$

However,

$$\hat{F}(u_0(\chi_R))=-\chi_R'(\tau)\operatorname{sech}^2 z. \tag{20}$$

Then we have

$$\hat{S}(u_0(\chi_R))=-\frac{2}{\sqrt{3}}\chi_R'(t). \tag{21}$$

Combine Eq. (17) with Eq. (21), we get the proto-RG equation as

$$\frac{d\chi_R}{dt}=-\varepsilon\frac{\sqrt{3}}{2}f_0=-\varepsilon\frac{3}{4}\int_{-\infty}^{+\infty}\hat{R}_1[u_0]\operatorname{sech}^2 z dz. \tag{22}$$

The important equation that determines how the soliton position is affected by the perturbation is also consistent with those derived by the other methods [10,11].

III. PROTO-RG THEORY FOR THE PERTURBED KdV EQUATION

Now we turn to the perturbed KdV equation

$$u_t+6uu_x+u_{xxx}=\varepsilon\hat{R}[u]. \tag{23}$$

The KdV equation

$$u_t+6uu_x+u_{xxx}=0 \tag{24}$$

has a single soliton solution such as

$$u=2A_0^2\operatorname{sech}^2 z,\quad z=A_0(x-\xi),\quad \xi=4A_0^2t+\chi_0, \tag{25}$$

where A_0 is the amplitude and χ_0 is the initial position of the soliton.

For convenience, we discuss it in terms of new variables $\{t,z\}$. To proceed, we construct a ε perturbation to this equation,

$$u=u_0(A_0,\chi_0)+\varepsilon u_1(t,z)+\cdots. \tag{26}$$

Here $u_0(A_0,\chi_0)=2A_0^2\operatorname{sech}^2 z$. Put it into Eq. (23), in the first order, we have

$$[\partial_t+A_0^3\hat{L}]u_1=\hat{R}_1[u_0], \tag{27}$$

where

$$\hat{L}=\frac{\partial^3}{\partial z^3}+(12\operatorname{sech}^2 z-4)\frac{\partial}{\partial z}-24\tanh z\operatorname{sech}^2 z. \tag{28}$$

Notice that there are two eigenfunctions of $\hat{L}$: one is $\Phi_0(z)=\tanh z\operatorname{sech}^2 z$, which satisfies $\hat{L}\Phi_0=0$, and the other is $\Phi_1(z)=(1-z\tanh z)\operatorname{sech}^2 z$, which satisfies $\hat{L}\Phi_1=-8\Phi_0$ [12].

To show the special structure of the operator $\hat{L}$ clearly, we can define the subspace (Ω) of $\hat{L}$ spanned by the degenerate zero eigenfunctions Φ_0 and Φ_1. In the null space the two basis vectors Φ_0 and Φ_1 are denoted as

$$\Phi_0=\begin{pmatrix}1\\0\end{pmatrix}\quad\text{and}\quad\Phi_1=\begin{pmatrix}0\\1\end{pmatrix}, \tag{29}$$

then the operator $\hat{L}$ can be expressed as the matrix

$$\hat{L}=-8\begin{pmatrix}0&1\\0&0\end{pmatrix}. \tag{30}$$

It is easy to validate the relations $\hat{L}\Phi_0=0$ and $\hat{L}\Phi_1=-8\Phi_0$.

We know the two degenerate zero eigenfunctions are the source of the secular terms in u_{1s}, and we can express u_{1s} in terms of the vectors Φ_0 and Φ_1,

$$u_{1s}=s_0\Phi_0+s_1\Phi_1, \tag{31}$$

where the s_0 and s_1 are the coefficients of components Φ_0 and Φ_1 of u_{1s}. Here u_{1s} is the singular term in u_1 and the regular terms are ignored. Now we can rewrite the first-order equation (27) in the null space (Ω) as

$$[\partial_t+A_0^3\hat{L}]\begin{pmatrix}s_0\\s_1\end{pmatrix}=\begin{pmatrix}f_0\\f_1\end{pmatrix}, \tag{32}$$

where f_0 and f_1 are the coefficients of components Φ_0 and Φ_1 of rhs of the first-order equation (27). Then we have the following equations:

$$\partial_t s_0=8A_0^3 s_1+f_0, \tag{33}$$

$$\partial_t s_1=f_1. \tag{34}$$

Before further calculation, we would like to point out that in Eq. (30) the original differential operator $\hat{L}$ has a ‘‘Jordan cell’’ in its matrix expression, and it is somewhat different from the simple structure of operator in Sec. II, which has only one zero eigenfucntion [see Eq. (13)], then a slight modification is necessary for the projection of the proto-RG operator as we will show in the following.

We can introduce the renormalized A_R and χ_R:

$$A_0 = Z_1(\tau)A_R = \left[1+\sum_1^{\infty} a_n\varepsilon^n\right]A_R,$$

$$\chi_0 = Z_2(\tau)\chi_R = \left[1+\sum_1^{\infty} b_n\varepsilon^n\right]\chi_R. \tag{35}$$

Then the Eq. (26) can be written as the renormalized perturbation result,

$$u = u_0(A_R,\chi_R) + \varepsilon[u_1(t) - u_{1s}(t,\tau)] + \cdots, \tag{36}$$

where the renormalization constants are chosen as

$$4A_R^2 a_1\Phi_1 + 4A_R^3(\chi_R b_1 + 8A_R^2 a_1 t)\Phi_0 + u_{1s}(t,\tau) = 0. \tag{37}$$

If we apply $\hat{S} = \hat{P}\hat{F}$ to u_{1s} (here $\hat{F} = \partial_\tau$), the operator $\hat{P}$ will contain two projections (which is slightly different from the case of only one projection in the present proto-RG method): one is on Φ_0 and the other is on Φ_1, which can be read from Eqs. (33) and (34). From them we find

$$\hat{S}u_{1s} = f_1 \tag{38}$$

for projection on Φ_1 and

$$\hat{S}u_{1s} = 8A_R^3 s_1 + f_0 \tag{39}$$

for projection on Φ_0. Here

$$f_i = \int_{-\infty}^{+\infty} \hat{R}_1[u_0]\Psi_i(z)dz, \tag{40}$$

i.e., this scalar product realizes the needed projection in the proto-RG operator, where $\Psi_0(z) = \tanh z + z\,\mathrm{sech}^2 z$ and $\Psi_1(z) = \mathrm{sech}^2 z$ are the adjoint function of $\Phi_0(z)$ and $\Phi_1(z)$.

We now apply the proto-RG operator $\hat{S}$ on the renormalized perturbation result (36), then we obtain

$$\hat{S}(u_0(A_R,\chi_R)) = \varepsilon\hat{S}u_{1s}. \tag{41}$$

However,

$$\hat{F}(u_0(A_R,\chi_R)) = 4A_R(\tau)A_R'(\tau)\Phi_1(z) + [32A_R^4(\tau)A_R'(\tau)t + 4A_R^3(\tau)\chi_R'(\tau)]\Phi_0(z). \tag{42}$$

Then we find

$$\hat{S}(u_0(A_R,\chi_R)) = 4A_R A_R' \tag{43}$$

for projection on Φ_1 and

$$\hat{S}(u_0(A_R,\chi_R)) = 32A_R^4 A_R' t + 4A_R^3\chi_R' \tag{44}$$

for projection on Φ_0.

Combine Eq. (38) with Eq. (43) corresponding to the component Φ_1 and Eq. (39) with Eq. (44) corresponding to the component Φ_0, we get the proto-RG equation as

$$\frac{dA_R}{dt} = \varepsilon\frac{1}{4A_R}f_1 = \varepsilon\frac{1}{4A_R}\int_{-\infty}^{+\infty} R_1[u_0]\mathrm{sech}^2 z\,dz, \tag{45}$$

$$\frac{d\chi_R}{dt} = \varepsilon\frac{1}{4A_R^3}f_0 = \varepsilon\frac{1}{4A_R^3}\int_{-\infty}^{+\infty} R_2[u_0](\tanh z + z\,\mathrm{sech}^2 z)dz, \tag{46}$$

which in this case consists of two independent equations.

The whole process gives the result

$$u(x,t) = 2A^2\,\mathrm{sech}^2 z, \quad A = A_R, \quad z = A(x-\xi), \quad \xi = 4A^2 t + \chi_R, \tag{47}$$

and

$$\frac{dA}{dt} = \varepsilon\frac{1}{4A}\int_{-\infty}^{+\infty} R[u_0]\mathrm{sech}^2 z\,dz, \tag{48}$$

$$\frac{d\xi}{dt} = 4A^2 + \varepsilon\frac{1}{4A^3}\int_{-\infty}^{+\infty} R[u_0](\tanh z + z\,\mathrm{sech}^2 z)dz. \tag{49}$$

The two important equations that determine how the soliton shape and position are affected by the perturbation are also consistent with those derived by IST [1,2].

As an example, we consider the damping KdV equation in which $R[u] = -u$. The time dependence of the soliton parameters can be easily obtained from Eqs. (48) and (49). Thus we have

$$A = A_0\exp\left(-\frac{2\varepsilon t}{3}\right), \tag{50}$$

$$\xi = \frac{3}{\varepsilon}A_0^2\left[1 - \exp\left(-\frac{4\varepsilon t}{3}\right)\right], \tag{51}$$

which is just the same as that obtained by IST [13].

Then we consider another example, the KdV-Burgers equation

$$u_t + 6uu_x + u_{xxx} = \varepsilon u_{xx}. \tag{52}$$

In the same way, we obtain

$$A = \frac{A_0}{\sqrt{1+\frac{16}{15}\varepsilon t A_0^2}}, \tag{53}$$

$$\xi = \frac{15}{4\varepsilon}\ln\left(1 + \frac{16}{15}\varepsilon t A_0^2\right), \tag{54}$$

which is also the same as that derived by IST [14].

IV. PROTO-RG THEORY FOR THE PERTURBED NLS EQUATION

In the above section, we have pointed out the special structure of the null subspace of the linearized first-order equation of the perturbed KdV equation and then suggested an extended proto-RG approach for it. Since the approach is a natural one from the RG point of view, we believe that this method is easy to follow and is used for perturbed NLS equation. Now we will give the details of the process.

We consider the perturbed NLS equation

$$iu_t+u_{xx}+2|u|^2u=i\varepsilon\hat{R}[u]. \tag{55}$$

The NLS equation

$$iu_t+u_{xx}+2|u|^2u=0 \tag{56}$$

has a single soliton solution such as

$$u=2\beta_0e^{-i\vartheta}\,\mathrm{sech}\,z,$$

$$z=2\beta_0(x-\xi),\quad \xi=-4\alpha_0t+\chi_0,$$

$$\vartheta=2\alpha_0(x-\xi)+\delta,\quad \delta=-4(\alpha_0^2+\beta_0^2)t+2\alpha_0\chi_0+\delta_0, \tag{57}$$

where β_0 is the amplitude, α_0 is the propagating velocity, χ_0 is the initial position, and δ_0 is the initial phase of the soliton.

We expand it in ε series and in terms of new coordinates $\{t,z\}$,

$$u=u_0(\alpha_0,\beta_0,\chi_0,\delta_0)+\varepsilon u_1(t,z)+\cdots. \tag{58}$$

Here $u_0(\alpha_0,\beta_0,\chi_0,\delta_0)=2\beta_0e^{-i\vartheta}\,\mathrm{sech}\,z$. Put it into Eq. (55), in the first order, we have

$$i\partial_tu_1+8i\alpha_0\beta_0\partial_zu_1+4\beta_0^2\partial_z^2u_1+4|u_0|^2u_1+2u_0^2\overline{u_1}]=\hat{R}_1[u_0], \tag{59}$$

where the overbar denotes the complex conjugate. To avoid the awkward calculation in complex plane, we can reduce it to coupled real equations by defining

$$u_1=e^{-i\vartheta}[A_1+iB_1],$$

where A_1 and B_1 are the real and imaginary parts of it. Now we can rewrite Eq. (59) into the following coupled real equations:

$$\partial_tA_1+4\beta_0^2\hat{L}_1B_1=\mathrm{Re}[\hat{R}_1e^{i\vartheta}], \tag{60}$$

$$\partial_tB_1-4\beta_0^2\hat{L}_2A_1=\mathrm{Im}[\hat{R}_1e^{i\vartheta}]. \tag{61}$$

Here

$$\hat{L}_1=\frac{\partial^2}{\partial z^2}+2\,\mathrm{sech}^2\,z-1,\quad \hat{L}_2=\frac{\partial^2}{\partial z^2}+6\,\mathrm{sech}^2\,z-1. \tag{62}$$

It is easy to derive that $\hat{L}_1\Phi_1=0$, $\hat{L}_1\Phi_2=-2\Psi_2$, $\hat{L}_2\Psi_2=0$, and $\hat{L}_2\Psi_1=2\Phi_1$ with $\Phi_1(z)=\mathrm{sech}\,z$, $\Phi_2(z)=z\,\mathrm{sech}\,z$, $\Psi1(\mathrm{z})=(1-\mathrm{z}\tanh z)\mathrm{sech}\,\mathrm{z}$, $\Psi2(\mathrm{z})=\tanh z\,\mathrm{sech}\,z$ [15].

Then we notice that $\hat{L}_1$ and $\hat{L}_2$ have the structures similar to the above section and we can construct a new operator $\hat{L}$ connecting with $\hat{L}_1$ and $\hat{L}_2$ and define the subspace (Ω) of $\hat{L}$ spanned by the four degenerate zero eigenfunctions Φ_1, Φ_2, Ψ_1, and Ψ_2. In the null space the four basis vectors Φ_1, Φ_2, Ψ_1, and Ψ_2 are denoted as

$$\Phi_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \Psi_1=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \Psi_2=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \Phi_2=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \tag{63}$$

where the operator $\hat{L}$ can be expressed as the matrix

$$\hat{L}=\begin{pmatrix}0&2&0&0\\0&0&0&0\\0&0&0&2\\0&0&0&0\end{pmatrix}. \tag{64}$$

It is easy to verify the relation $\hat{L}\Phi_1=0$, $\hat{L}\Psi_1=2\Phi_1$, $\hat{L}\Psi_2=0$, and $\hat{L}\Phi_2=2\Psi_2$.

We know these four degenerate zero eigenfunctions are the source of the secular terms in u_{1s}, and we can assume that the expression, i.e., expand u_{1s} in the null space with their basis vectors:

$$u_{1s}=e^{-i\vartheta}[A_{1s}+iB_{1s}]=e^{-i\vartheta}[s_1\Psi_1+s_2\Psi_2+iv_1\Phi_1+iv_2\Phi_2]. \tag{65}$$

Now we rewrite the coupled equations (60) and (61) in the null space (Ω) as

$$[\partial_t+4\beta_0^2\hat{L}]\begin{pmatrix}v_1\\v_2\\s_1\\s_2\end{pmatrix}=\begin{pmatrix}g_1\\g_2\\f_1\\f_2\end{pmatrix}, \tag{66}$$

where the f_1,f_2 are the coefficients of components Ψ_1 and Ψ_2 of rhs of the first-order equation (60) and the g_1,g_2 are the coefficients of components Φ_1 and Φ_2 of rhs of the first-order equation (61). Then we have the following equations:

$$\partial_tv_1=8\beta_0^2s_1+g_1, \tag{67}$$

$$\partial_tv_2=g_2, \tag{68}$$

$$\partial_t s_1 = f_1, \tag{69}$$

$$\partial_t s_2 = 8\beta_0^2 v_2 + f_2. \tag{70}$$

Before further calculation, we would like to point out that in Eq. (64), the original differential operators $\hat{L}_1$ and $\hat{L}_2$ have a four dimensional "Jordan cell" in their matrix expression $\hat{L}_1$, similar to the structure of Eq. (30) in the case of the KdV equation. Then a treatment similar to that in Sec. III can be performed in the following.

Then introduce the renormalized parameters as

$$\alpha_R = Z_1^{-1}(\tau)\alpha_0, \quad \beta_R = Z_2^{-1}(\tau)\beta_0, \quad \chi_R = Z_3^{-1}(\tau)\chi_0,$$
$$\delta_R = Z_4^{-1}(\tau)\delta_0, \tag{71}$$

$$Z_1 = 1 + \sum_1^\infty a_n\varepsilon^n, \quad Z_2 = 1 + \sum_1^\infty b_n\varepsilon^n,$$

$$Z_3 = 1 + \sum_1^\infty c_n\varepsilon^n, \quad Z_4 = 1 + \sum_1^\infty d_n\varepsilon^n, \tag{72}$$

and we have the following renormalized perturbation result:

$$u = u_0(\alpha_R, \beta_R, \chi_R, \delta_R) + \varepsilon[u_1(t) - u_{1s}(t,\tau)] + \cdots, \tag{73}$$

where the renormalization constants are chosen as

$$e^{-i\vartheta}[2b_1\beta_R\Psi_1 - 4\beta_R^2(4a_1\alpha_R t - c_1\chi_R)\Psi_2 - i(4a_1\alpha_R\beta_R\chi_R - 16b_1\beta_R^3 t + 2d_1\beta_R\delta_R)\Phi_1 - i2a_1\alpha_R\Phi_2] + u_{1s}(t,\tau) = 0. \tag{74}$$

If we apply $\hat{S} = \hat{P}\hat{F}$ to u_{1s} (here $\hat{F} = \partial_\tau$), the operator $\hat{P}$ will contain four projections: on Φ_1, Φ_2, and on Ψ_1, Ψ_2, which can be read from Eqs. (67)–(70). From them we get

$$\hat{S}u_{1s} = 8\beta_R^2 s_1 + g_1 \tag{75}$$

for the projection on Φ_1,

$$\hat{S}u_{1s} = g_2 \tag{76}$$

for the projection on Φ_2,

$$\hat{S}u_{1s} = f_1 \tag{77}$$

for the projection on Ψ_1,

$$\hat{S}u_{1s} = 8\beta_R^2 v_2 + f_2 \tag{78}$$

for the projection on Ψ_2. Here

$$f_i = \int_{-\infty}^{+\infty} \mathrm{Re}[\hat{R}_1 e^{i\vartheta}]\Phi_i(z)dz,$$

$$g_i = \int_{-\infty}^{+\infty} \mathrm{Im}[\hat{R}_1 e^{i\vartheta}]\Psi_i(z)dz, \tag{79}$$

i.e., this scalar product realizes the needed projection in the proto-RG operator.

We now apply the proto-RG operator $\hat{S}$ on the renormalized perturbation result (73), then we obtain

$$\hat{S}(u_0(\alpha_R, \beta_R, \chi_R, \delta_R)) = \varepsilon\hat{S}u_{1s}. \tag{80}$$

However,

$$\hat{F}u_0 = \exp(-i\vartheta)[2\beta_R'\Psi_1 - 4\beta_R^2(4\alpha_R' t - \chi_R')\Psi_2 - i(4\alpha_R'\beta_R\chi_R - 16\beta_R^2\beta_R' t + 2\beta_R\delta_R')\Phi_1 - 2i\alpha_R'\Phi_2]. \tag{81}$$

Then we have

$$\hat{S}u_0 = -(4\alpha_R'\beta_R\chi_R - 16\beta_R^2\beta_R' t + 2\beta_R\delta_R') \tag{82}$$

for projection on Φ_1,

$$\hat{S}u_0 = -2\alpha_R' \tag{83}$$

for projection on Φ_2,

$$\hat{S}u_0 = 2\beta_R' \tag{84}$$

for projection on Ψ_1,

$$\hat{S}u_0 = -4\beta_R^2(4\alpha_R' t - \chi_R') \tag{85}$$

for projection on Ψ_2.

Combine Eqs. (75)–(78) with Eqs. (82)–(85) corresponding to each component of Ω, we get the proto-RG equation as

$$\frac{d\alpha_R}{dt} = -\varepsilon\frac{1}{2}g_2, \tag{86}$$

$$\frac{d\beta_R}{dt} = \varepsilon\frac{1}{2}f_1, \tag{87}$$

$$\frac{d\chi_R}{dt} = \varepsilon\frac{f_2}{4\beta_R^2}, \tag{88}$$

$$\frac{d\delta_R}{dt} = -\varepsilon\frac{g_1}{2\beta_R} + \varepsilon\chi_R g_2, \tag{89}$$

which in this case consists of four independent equations.

Finally, the above process gives the result

$$u(x,t) = 2\beta e^{-i\vartheta}\,\mathrm{sech}\, z,$$
$$\beta = \beta_R, \quad \alpha = \alpha_R, \quad z = 2\beta(x - \xi), \quad \xi = -4\alpha t + \chi_R,$$
$$\vartheta = 2\alpha(x - \xi) + \delta, \quad \delta = -4(\alpha^2 + \beta^2)t + 2\alpha\chi_R + \delta_R \tag{90}$$

and

$$\frac{d\alpha}{dt} = -\varepsilon\frac{1}{2}g_2 = -\varepsilon\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{Im}[\hat{R}_1 e^{i\vartheta}]\tanh z\,\mathrm{sech}\, z dz, \tag{91}$$

$$\frac{d\beta}{dt} = \varepsilon\frac{1}{2}f_1 = \varepsilon\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{Re}[\hat{R}_1 e^{i\vartheta}]\mathrm{sech}\, z dz, \tag{92}$$

$$\frac{d\xi}{dt}=-4\alpha+\varepsilon\frac{f_2}{4\beta^2}$$

$$=-4\alpha+\varepsilon\frac{1}{4\beta^2}\int_{-\infty}^{+\infty}\mathrm{Re}[\hat{R}_1 e^{i\vartheta}]z\,\mathrm{sech}\,z dz, \qquad (93)$$

$$\frac{d\delta}{dt}=4(\alpha^2-\beta^2)+2\alpha\frac{d\xi}{dt}$$

$$-\varepsilon\frac{1}{2\beta}\int_{-\infty}^{+\infty}\mathrm{Im}[\hat{R}_1 e^{i\vartheta}](1-z\tanh z)\mathrm{sech}\,z dz. \qquad (94)$$

The two important equations that determine how the soliton shape and position are affected by the perturbation are also consistent with those derived by IST [1,16].

V. CONCLUSION

In summary, we have demonstrated that a perturbed KdV equation can be solved by a proto-RG method, with some attendant technical advantages compared with the other methods. The present approach can easily be generalized also to multiple-soliton state and to the soliton-soliton interaction. To avoid unnecessary complications, we expound our theory using as example the three equations: KdV, NLS, and TDGL. It is, however, clear from what follows that this does not restrict the general nature of the method for another perturbed evolution equations.

On the other hand, the present proto-RG method [8,9] demonstrates its principle with various examples focusing on the case of only one zero eigenfunction in the first-order equation, however, in this paper we apply it to include the case of degenerate zero eigenfucntions, which has more than one source for the secular terms, then when we use the proto-RG operator on u_{1s}; it should be classified into the different projection on each zero eigenfucntion and the resulting in proto-RG equation can be read from each one easily. The standard RG version of soliton perturbation can be found in Ref. [17], whereas they do not obtain the the second RG equation (49) for KdV because they concentrate on only one zero eigenfunction Φ_0 and neglect the discussion about the other degenerate eigenfunction Φ_1.)

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